Math 121 2.7 Nondifferentiable Functions

Objectives

1) Use the words "differentiable" and "nondifferentiable" correctly
2) Recall
   - definition of derivative (2.2)
   - reasons a limit does not exist (2.1)
3) Use the definition of the derivative to demonstrate that a given function is not differentiable at a given value of x.
4) Use the graph of a function to identify the values of x where the derivative does not exist.
5) Understand the relationship between continuity (2.1) and differentiability (2.2).

Recall:

(A) definition of derivative of f(x)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ = slope of tangent line to f(x) at the point (x, f(x))

(B) reasons a limit could not exist

(i) $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$ — leftside limit $\neq$ rightside limit

(ii) $\lim_{x \to c} f(x) = \infty$ or $-\infty$ — y-coordinate approaches infinity.

Put together reasons the limit (defn of derivative) does not exist:

(A)+(B) means

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ could not exist

if

(i) $\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \neq \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$ — slope of tangent on left $\neq$ slope of tangent on right

or (ii) $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \infty$ or $-\infty$. — slope of tangent is ∞

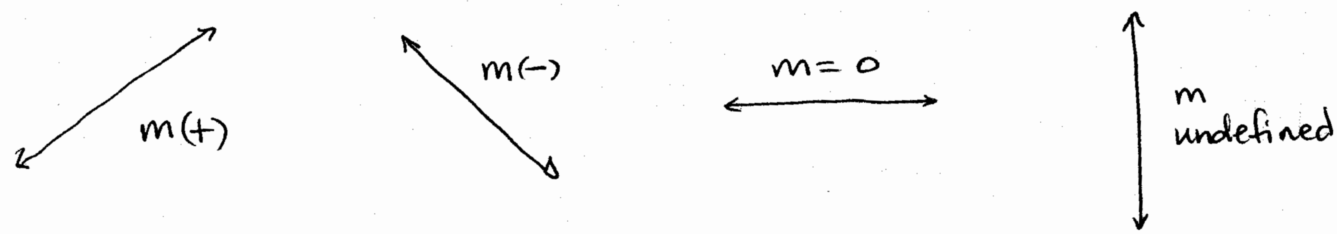
So if we want to know if the derivative of a function $f(x)$ exists at a value of $x=c$, we are asking

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

- is the same value as $h\to 0^+$ as if $h\to 0^-$?
- is finite ?

So what do we mean by $h\to 0^+$ and $h\to 0^-$. ?
And what do we mean by "infinite slope of tangent line?

<u>Recall slope of any line</u>:



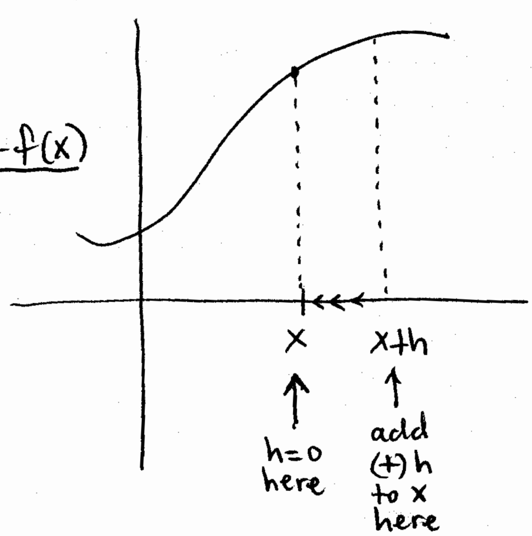
↑
This is what we will now call <u>infinite slope</u>.

So if the tangent line at $x=c$ has infinite slope

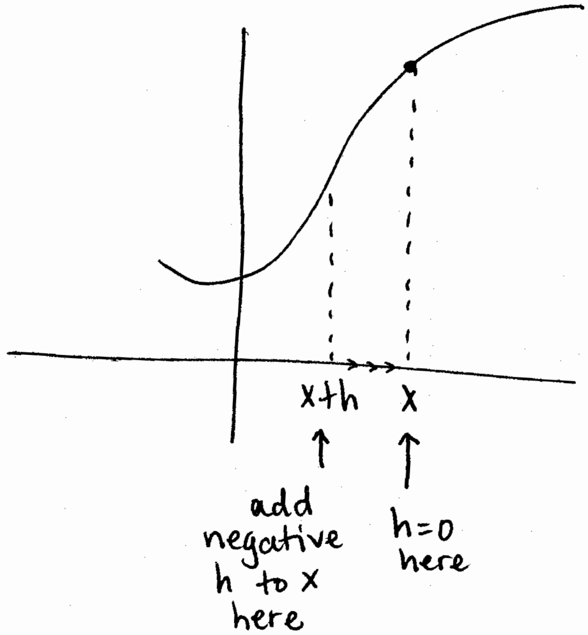
- the tangent line is vertical at $x=c$
- the $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \pm\infty$ does not exist
- $f(x)$ is not differentiable at $x=c$.
  or. <u>nondifferentiable</u>

<u>Recall definition of derivative</u>:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

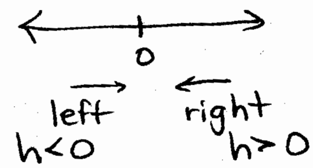


This picture implies that h is positive!
As $h\to 0^+$, this picture applies.



$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

This limit must work if we approach 0 from left or right



This picture implies that h is negative!
As $h \to 0^-$, this picture applies.

① Use the definition of the derivative to demonstrate that $f(x)=|x|$ is nondifferentiable at $x=0$.

Definition of derivative $\qquad f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Substitute $f(x)=|x|$ $\qquad f'(x) = \lim_{h \to 0} \frac{|x+h|-|x|}{h}$

Substitute $x=0$ $\qquad f'(0) = \lim_{h \to 0} \frac{|0+h|-|0|}{h}$

$$f'(0) = \lim_{h \to 0} \frac{|h|}{h}$$

Consider limits from left and right separately:

$$\lim_{h \to 0^+} \frac{|h|}{h} = 1$$

Table

h	$\lvert h\rvert/h$
.1	1
.01	1
.001	1
.0001	1
.00001	1

$$\frac{|.001|}{.001} = \frac{.001}{.001} = +1$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = -1$$

Table

h	$\frac{\lvert h\rvert}{h}$
$-.1$	-1
$-.01$	-1
$-.001$	-1
$-.0001$	-1
$-.00001$	-1

$$\frac{|-.001|}{-.001} = \frac{.001}{-.001} = -1$$

So $f'(0)$ does not exist because

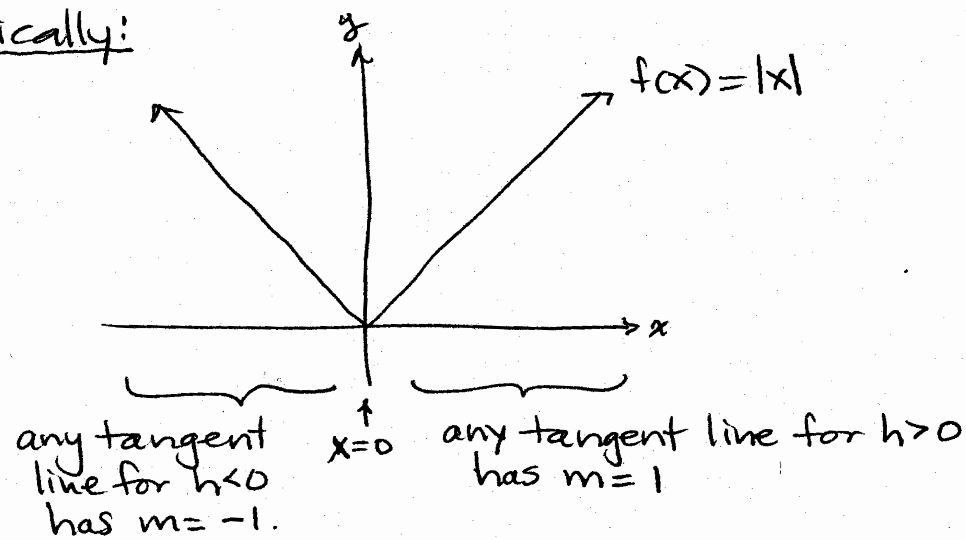
$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = +1$$

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = -1$$

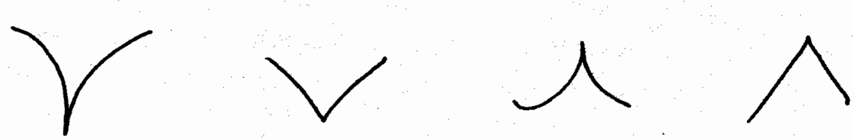
not the same value!

$f(x) = |x|$ is nondifferentiable at $x=0$.

Graphically:



This shape is called a corner or cusp.



Any time a graph has a corner or cusp, it is not differentiable at that value of x.

② Use the definition of the derivative to demonstrate that $f(x) = \sqrt[3]{x}$ is nondifferentiable at $x=0$.

Definition of derivative $\quad f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

Substitute $f(x) = \sqrt[3]{x}$ $\quad f'(x) = \lim\limits_{h\to 0} \dfrac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h}$

Substitute $x=0$ $\quad f'(0) = \lim\limits_{h\to 0} \dfrac{\sqrt[3]{0+h} - \sqrt[3]{0}}{h}$

$$f'(0) = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h}$$

Consider limits from left and right separately:

$\lim\limits_{h\to 0^+} \dfrac{\sqrt[3]{h}}{h} = +\infty$ Table

h	$\frac{\sqrt[3]{h}}{h}$
.1	4.6
.01	21.5
.001	100
.0001	464.2
.00001	2154.4
.000001	10 000

$\dfrac{\sqrt[3]{.1}}{.1} \approx 4.6416$

$\dfrac{\sqrt[3]{.01}}{.01} \approx 21.544$

These values are approaching $+\infty$!!

$\lim\limits_{h\to 0^-} \dfrac{\sqrt[3]{h}}{h}$

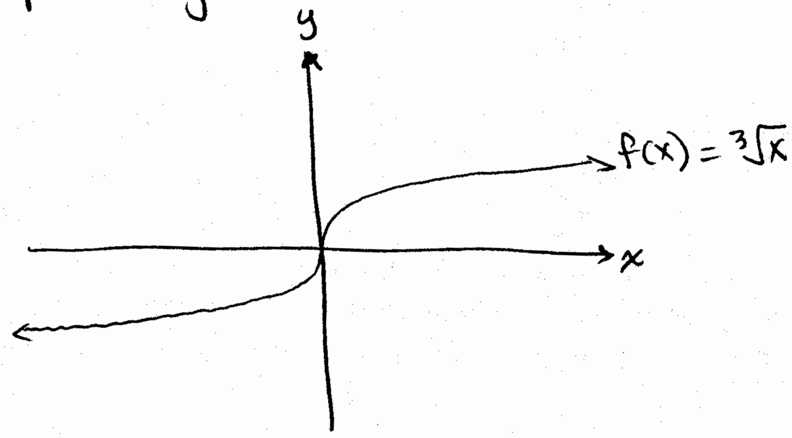
h	$\frac{\sqrt[3]{h}}{h}$
-.1	+4.6
-.01	+21.5
-.001	+100
-.0001	+464.2
-.00001	+2154.4
-.000001	+10000

$\dfrac{\sqrt[3]{-.1}}{-.1} \approx 4.6$

These values also approach $+\infty$!!

So $f'(0)$ does not exist because $f'(0) = +\infty$.
$f(x) = \sqrt[3]{x}$ is nondifferentiable at $x = 0$.

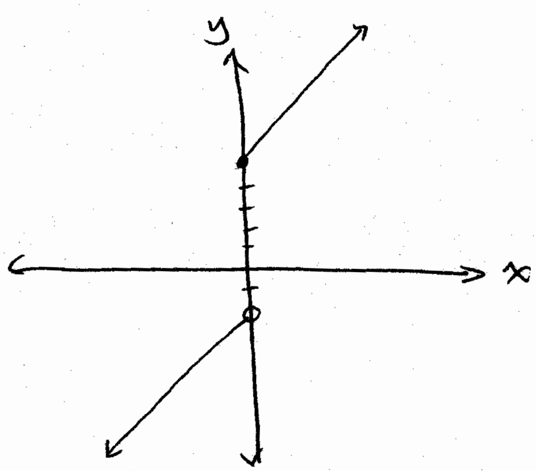
Graphically.



tangent line to $f(x) = \sqrt[3]{x}$ at $x = 0$ is vertical.

③ Graph the piecewise function

$$f(x) = \begin{cases} x+5 & x \geq 0 \\ x-2 & x < 0 \end{cases}$$



④ Is $f(x) = \begin{cases} x+5 & x \geq 0 \\ x-2 & x < 0 \end{cases}$ differentiable at $x = 0$?

Use the definition of the derivative to demonstrate.

Definition of derivative $\quad f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

subst $x = 0$ $\quad f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$f(0) = 5$ always!
• on graph

$f'(0) = \lim\limits_{h \to 0} \dfrac{f(h) - 5}{h}$

Consider limits from the left and right separately.

$$\lim_{h\to 0^+} \frac{f(h)-5}{h}$$ means $h>0$ so $(x+5)$ is applicable

$$= \lim_{h\to 0^+} \frac{h+5-5}{h}$$

$$= \lim_{h\to 0^+} \frac{h}{h}$$

$$= \lim_{h\to 0^+} 1$$

$$= 1$$ limit from the right is 1.

$$\lim_{h\to 0^-} \frac{f(h)-5}{h}$$ means $h<0$ so $x-2$ is applicable but $f(0)=5$ still!

$$= \lim_{h\to 0^-} \frac{h-2-5}{h}$$

$$= \lim_{h\to 0^-} \frac{h-7}{h}$$

$$= +\infty$$

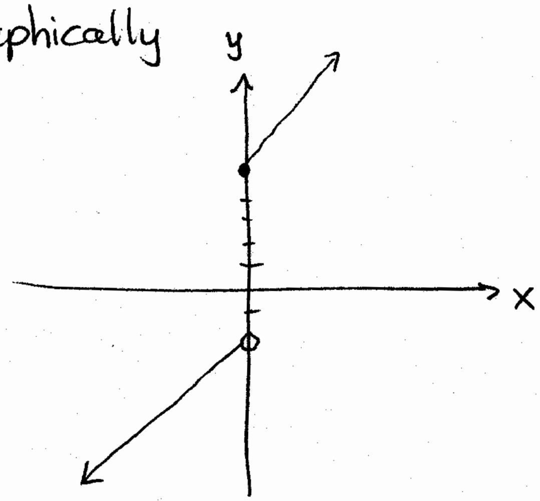
Table

h	$\frac{h-7}{h}$
$-.1$	71
$-.01$	701
$-.001$	7001
$-.0001$	70001
$-.00001$	700001

approaching $+\infty$

$f'(0)$ does not exist for <u>two</u> reasons
1) limit from left $\neq$ limit from right
2) limit from left is infinite.

Graphically



The function

$$f(x)=\begin{cases} x+5 & x\geq 0 \\ x-2 & x<0 \end{cases}$$

is not continuous at $x=0$ because we must pick up the pencil at $x=0$ when drawing the graph. $\lim_{x\to c} f(x) \neq f(c)$.

Functions which are not continuous at $x=c$ are also not differentiable at $x=c$.

Summary of Graphical Features of Nondifferentiability

1) $f'(c)$ does not exist if there is a cusp or corner at $x=c$.

2) $f'(c)$ does not exist if there is a vertical tangent line at $x=c$.

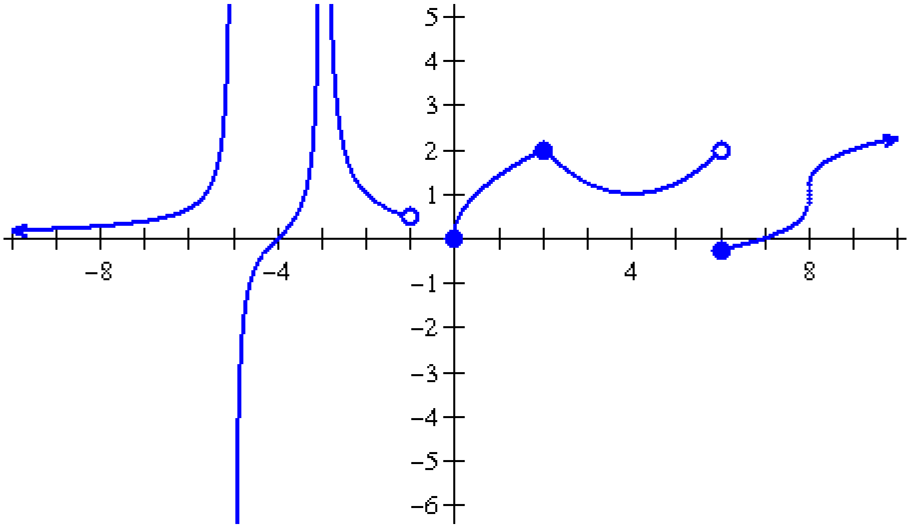
3) $f'(c)$ does not exist if $f(x)$ is discontinuous at $x=c$.

Note: Both $f(x)=|x|$ and $f(x)=\sqrt[3]{x}$ are continuous at $x=0$ but not differentiable at $x=0$.

If a function is continuous, it could be either differentiable or nondifferentiable.

If a function is differentiable, it must be continuous.

5) Identify the values of x where the function shown in the graph is nondifferentiable.



All of the following are locations of non-differentiability:

$x = -5$ is a vertical asymptote. Neither $f(-5)$nor $f'(-5)$ is defined.

$x = -3$ is a vertical asymptote. Neither $f(-3)$nor $f'(-3)$ is defined.

For $-1 \le x < 0$, either $f(x)$ nor $f'(x)$ is defined.

There is a cusp at $x = 2$.

There is a discontinuity at $x = 6$

There is a vertical tangent at $x = 8$